

Stabilizing on the distinguishing number of a graph

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Abstract

The distinguishing number $D(G)$ of a graph G is the least integer d such that G has a vertex labeling with d labels that is preserved only by a trivial automorphism. The distinguishing stability, of a graph G is denoted by $st_D(G)$ and is the minimum number of vertices whose removal changes the distinguishing number. We obtain a general upper bound $st_D(G) \leq |V(G)| - D(G) + 1$, and a relationships between the distinguishing stabilities of graphs G and $G - v$, i.e., $st_D(G) \leq st_D(G - v) + 1$, where $v \in V(G)$. Also we study the edge distinguishing stability number (distinguishing bondage number) of G .

Keywords: distinguishing number; stability; bondage number

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1 Introduction and definitions

Let $G = (V, E)$ be a simple graph of order $n \geq 2$. We use the the following notations: The set of vertices adjacent in G to a vertex of a vertex subset $W \subseteq V$ is the open neighborhood $N_G(W)$ of W . The closed neighborhood $G[W]$ also includes all vertices of W itself. In case of a singleton set $W = \{v\}$ we write $N_G(v)$ and $N_G[v]$ instead of $N_G(\{v\})$ and $N_G[\{v\}]$, respectively. $\text{Aut}(G)$ denotes the automorphism group of G . A labeling of G , $\phi : V \rightarrow \{1, 2, \dots, r\}$, is said to be r -distinguishing, if no non-trivial automorphism of G preserves all of the vertex labels. The point of the labels on the vertices is to destroy the symmetries of the graph, that is, to make the automorphism group of the labeled graph trivial. Formally, ϕ is r -distinguishing if for every non-trivial $\sigma \in \text{Aut}(G)$, there exists x in V such that $\phi(x) \neq \phi(x\sigma)$. The distinguishing number of a graph G is defined by

$$D(G) = \min\{r \mid G \text{ has a labeling that is } r\text{-distinguishing}\}.$$

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This number has defined by Albertson and Collins [1]. If a graph has no nontrivial automorphisms, its distinguishing number is one. In other words, $D(G) = 1$ for the asymmetric graphs. The other extreme, $D(G) = |V(G)|$, occurs if and only if $G = K_n$. The distinguishing number of some examples of graphs, $D(P_n) = 2$ for every $n \geq 3$, and $D(C_n) = 3$ for $n = 3, 4, 5$, $D(C_n) = 2$ for $n \geq 6$. Also $D(K_{p,q}) = p$, for $p > q$, and $D(K_{p,p}) = p + 1$, for $p \geq 4$. Authors in [3] have shown that removing a vertex of G can decrease the distinguishing number by at most one but can increase by at most to double of distinguishing number of G . Also for each connected graph G and $e \in E(G)$, $|D(G - e) - D(G)| \leq 2$.

A domination-critical (domination-super critical, respectively) vertex in a graph G is a vertex whose removal decreases (increases, respectively) the domination number. Bauer et al. [5] introduced the concept of domination stability in graphs. The domination stability, or just γ -stability, of a graph G is the minimum number of vertices whose removal changes the domination number. Motivated by domination stability, we introduce the distinguishing stability of a graph.

Definition 1.1 *Let G be a graph of order $n \geq 2$. The stabilizing on the distinguishing number, or just distinguishing stability, $st_D(G)$ of graph G is the minimum number of vertices whose removal changes the distinguishing number.*

Also we introduce and study the edge distinguishing stability number (distinguishing bondage number) of G and compute edge distinguishing stability of some specific graphs.

In the next section we compute the distinguishing stability of some specific graphs. We obtain general bounds, and a relationships between the distinguishing stabilities of G and $G - v$, where $G - v$ denotes the graph obtained from G by removal of a vertex v and all edges incident to v , in Section 3. Finally we consider and study the edge distinguishing stability number of graphs in Section 4.

2 Distinguishing stability of specific graphs

In this section, first we compute the distinguishing stability of some specific graphs. We start with paths and cycles. A path is a connected graph in which two vertices have degree one and the remaining vertices have degree two. Let P_n be the path with n vertices as shown in Figure 1.

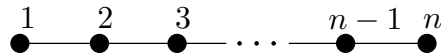


Figure 1: A path P_n with vertex set $\{1, \dots, n\}$.

Proposition 2.1 *For any $n \geq 6$, the distinguishing stability of P_n , is $st_D(P_n) = 2$, while $st_D(P_2) = 1$, $st_D(P_3) = 2$, $st_D(P_4) = 3$, and $st_D(P_5) = 1$.*

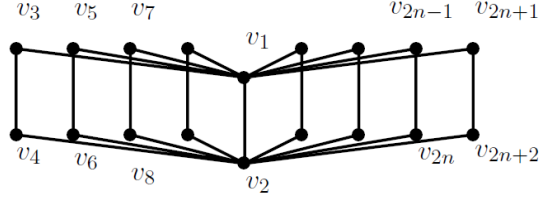


Figure 2: Book graph B_n .

Proof. It is clear that $st_D(P_n) > 1$ for $n \geq 6$. On the other hand by removing the third and sixth vertex of P_n , i.e., vertices labeled by numbers 3 and 6, the graph P_n convert to disjoint union of the two paths P_2 and one path P_{n-5} , and so $D(P_n - \{3, 6\}) = 3$. Since $D(P_n) = 2$, so $st_D(P_n) = 2$ for $n \geq 6$. \square

The following proposition obtain immediately from Proposition 2.1.

Proposition 2.2 *For any $n \geq 7$, the distinguishing stability of C_n is, $st_D(C_n) = 3$, while $st_D(C_3) = st_D(C_4) = st_D(C_5) = 1$, and $st_D(C_6) = 2$.*

With respect to the value of the distinguishing number of the complete graphs and complete bipartite graphs we can prove the following proposition:

Proposition 2.3 (i) *For any $p \geq 1$, $st_D(K_p) = 1$.*

(ii) *For any $n \geq m$, $st_D(K_{n,m}) = 1$, except, $st_D(K_{n,n+1}) = 2$.*

The n -book graph ($n \geq 2$) (Figure 2) is defined as the Cartesian product $K_{1,n} \square P_2$. We call every C_4 in the book graph B_n , a page of B_n . All pages in B_n have a common side v_1v_2 . We shall compute the distinguishing stability number of B_n . The following theorem gives the distinguishing number of the book graph.

Theorem 2.4 [2] *The distinguishing number of B_n ($n \geq 2$) is $D(B_n) = \lceil \sqrt{n} \rceil$.*

Proposition 2.5 *The distinguishing stability of the book graph is*

$$st_D(B_n) = \begin{cases} 1 & \text{if } n-1 \text{ is square,} \\ 2 & \text{otherwise.} \end{cases}$$

Proof. By removing the two central vertices of B_n , the book graph B_n convert to disjoint union of n paths P_2 , denoted by nP_2 . It can be computed that the distinguishing number of nP_2 is $D(nP_2) = \lceil \frac{1 + \sqrt{8n+1}}{2} \rceil$ which is different from the distinguishing number of B_n , $D(B_n) = \lceil \sqrt{n} \rceil$. So $st_D(B_n) \leq 2$. If the value of $n-1$ is square, then $D(B_n) = D(B_{n-1}) + 1$, and so by removing a noncentral vertex of B_n , say v , we have a book graph B_{n-1} and the path P_3 , that start point of P_3 is identified with one of the central point of B_{n-1} , and so P_3 is fixed by any automorphism of $B_n - v$. So $D(B_n - v) = D(B_{n-1})$, and hence $st_D(B_n) = 1$. \square

The friendship graph F_n ($n \geq 2$) can be constructed by joining n copies of the cycle graph C_3 with a common vertex. To compute the distinguishing stability of F_n , we need the following theorem.

Theorem 2.6 [2] *The distinguishing number of the friendship graph F_n ($n \geq 2$) is*

$$D(F_n) = \lceil \frac{1 + \sqrt{8n+1}}{2} \rceil.$$

Proposition 2.7 *The distinguishing stability of the friendship graph is*

$$st_D(F_n) = \min\{k : 8(n-k) + 1 \text{ is square}\}.$$

Proof. Set $\min\{k : 8(n-k) + 1 \text{ is square}\} = t$ and consider the graph F_n as shown in Figure 3. The graph $F_n - \{v_1, v_3, \dots, v_{2t-1}\}$ is two graphs F_{n-t} and $K_{1,t}$ such that their central vertices are identified. Since t is the minimum number which $8(n-t) + 1$ is square, so $D(F_{n-t}) = D(F_n) - 1$. On the other hand, since $K_{1,t}$ and F_{n-t} are two nonisomorphic graphs, so $D(F_n - v_1 - v_3 - \dots - v_{2t-1}) = \max\{D(F_{n-t}), D(K_{1,t})\}$, hence $D(F_n - v_1 - v_3 - \dots - v_{2t-1}) = D(F_n) - 1$. Thus $st_D(F_n) \leq t$. Now we show that $st_D(F_n) > t - 1$. First note that if we remove the central vertex of F_n , say w , then the value of the distinguishing number of $F_n - w$ is equal $D(F_n)$. In general, if v_{i_1}, \dots, v_{i_s} are s noncentral vertices of F_n , then $D(F_n - v_{i_1} - \dots - v_{i_s}) = D(F_n - v_{i_1} - \dots - v_{i_s} - w)$, because the central vertex w is fixed under each automorphism and $\text{Aut}(F_n - v_{i_1} - \dots - v_{i_s}) \cong \text{Aut}(F_n - v_{i_1} - \dots - v_{i_s} - w)$.

By using above point and regarding to the value of t , if we remove less than t vertices of F_n , say $v_{i_1}, \dots, v_{i_{t-1}}$, then $D(F_n - v_{i_1} - \dots - v_{i_{t-1}}) = D(F_n)$, and so $st_D(F_n) > t - 1$. Therefore $D(F_n) = t$. \square

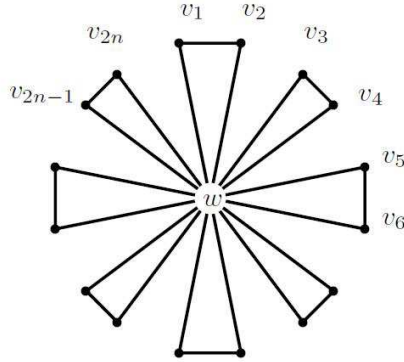


Figure 3: Friendship graph F_n .

With respect to the value of $st_D(F_n)$ for different values of n we can prove the following result:

Corollary 2.8 *For every natural number k , there exists a graph G such that $st_D(G) = k$.*

Proof. If $k = 1$, then it is suffice to consider the complete graphs. For $k \geq 2$ the result follows immediately from Proposition 2.7 \square

3 Upper bounds for the distinguishing stability

In this section, we study the relationship between the distinguishing stabilities of graphs G and $G - v$, where $v \in V(G)$. Also we obtain upper bounds for $st_D(G)$.

Proposition 3.1 *Let G be a graph and v be a vertex of G , then*

$$st_D(G) \leq st_D(G - v) + 1.$$

Proof. If $D(G) = D(G - v)$, then $st_D(G) \leq st_D(G - v) + 1$. If $D(G) \neq D(G - v)$, then $st_D(G) = 1$, and so we have the result. \square

Using Proposition 3.1 and mathematical induction, we have $st_D(G) \leq st_D(G - v_1 - \dots - v_s) + s$ where $1 \leq s \leq n - 2$ and $n = |V(G)|$. Only using this formula we can get different upper bounds for $st_D(G)$ of graph G . We state some these upper bounds in the following theorem which for the prove of each case, it is sufficient to remove the vertices of G until the induced subgraph which stated in the hypothesis, appears. Next using $st_D(G) \leq st_D(G - v_1 - \dots - v_s) + s$ and the value of $st_D(G - v_1 - \dots - v_s)$, we can have the result.

Theorem 3.2 *Let G be a simple graph of order $n \geq 2$.*

(i) $st_D(G) \leq n - 1$.

(ii) *If Δ is maximum degree of graph G , and G has the star graph $K_{1,\Delta}$ as the induced subgraph with $\Delta \geq 3$, then $st_D(G) \leq n - \Delta$.*

(iii) *If d is the diameter of graph G , and G has the path P_{d+1} as the induced subgraph, then*

$$st_D(G) \leq \begin{cases} n - d + 2 & d = 3, \\ n - d & d = 1, 4, \\ n - d + 1 & \text{otherwise.} \end{cases}$$

(iv) $st_D(G) \leq n - \omega(G) + 1$, where $\omega(G)$ is the clique number of G .

Theorem 3.3 *Let G be a graph of order n , then $st_D(G) \leq n - D(G) + 1$.*

Proof. Let $st_D(G) = k$. So for every i ($1 \leq i \leq k - 1$) vertices of G , say v_1, \dots, v_{k-1} , we have $D(G) = D(G - v_1) = \dots = D(G - v_1 - \dots - v_{k-1})$. Since the distinguishing number of a graph is at most equal to its order, so $D(G) = D(G - v_1 - \dots - v_{k-1}) \leq n - k + 1$. \square

A graph and its complement, always have the same automorphism group while their graph structure usually differs. Hence $D(G) = D(\overline{G})$ for every simple graph G . In the following theorem we use this equality to show that the distinguishing stability of a graph and its complement are the same.

Theorem 3.4 *Let G be a simple graph, then $st_D(G) = st_D(\overline{G})$.*

Proof. Since $D(G) = D(\overline{G})$ and $\overline{G} - v_1 - \dots - v_k = \overline{G - v_1 - \dots - v_k}$, we have the result. \square

Theorem 3.5 *If there exists a vertex v of G such that $st_D(G) \leq st_D(G - v)$, then $st_D(G) \leq D(G)$.*

Proof. If $D(G) \neq D(G - v)$, then $st_D(G) = 1$, and so $st_D(G) \leq D(G)$. If $D(G) = D(G - v)$, then we use mathematical induction on the order of G . It can be seen that the result is true for small value of n . Let $st_D(G) \leq D(G)$ for all graphs of order $n < k$. Suppose that $n = k$, in this case by induction hypothesis we have $st_D(G - v) \leq D(G - v)$. Thus we can write

$$st_D(G) \leq st_D(G - v) \leq D(G - v) = D(G).$$

Therefore the result follows. \square

We think that for any simple graph G the inequality $st_D(G) \leq D(G) + 1$ is true. However, until now all attempts to find a proof failed. So we propose the following conjecture here.

Conjecture 3.6 *Let G be a simple connected graph, then $st_D(G) \leq D(G) + 1$.*

Let v be a vertex in G . The contraction of v in G denoted by G/v is the graph obtained by deleting v and putting a clique on the (open) neighbourhood of v . Note that this operation does not create parallel edges; if two neighbours of v are already adjacent, then they remain simply adjacent (see [6]). In the end of this section, we study the distinguishing stabilities of graph G/v .

Theorem 3.7 *Let G be a graph and v be a vertex of it, and let e_1, \dots, e_k be the added edges to the neighbours of v in the construction of G/v . Suppose that v_1, \dots, v_t are all different neighbours of v which are incident to at least one of the edges e_1, \dots, e_k . We have*

$$st_D(G/v) \leq st_D(G - v - v_1 - \dots - v_t) + t.$$

Proof. First of all note that if e is an edge incident to the vertex w of G , then by Proposition 3.1, $st_D(G - e) \leq st_D(G - w) + 1$. Using Theorem 3.4 we can write

$$st_D(G/v) = st_D(G - v + e_1 + \dots + e_k) = st_D(\overline{G} - v - e_1 - \dots - e_k).$$

By the first sentence of proof, it can be concluded that

$$st_D(\overline{G} - v - e_1 - \dots - e_k) \leq st_D(\overline{G} - v - v_1 - \dots - v_t) + t.$$

Now using Theorem 3.4, we obtain the result. \square

4 Distinguishing bondage number of a graph

In this section we study the edge stability (bondage) number on the distinguishing number of a graph. The bondage number on the distinguishing number $b_D(G)$, (or the edge distinguishing stability number) is the minimum number of edges whose removal changes the distinguishing number. First we determine $b_D(G)$ for several families of graphs including complete (bipartite) graphs, cycles, and paths.

Proposition 4.1 (i) For $n \geq 3$ and $m \geq 2$, we have $b_D(K_n) = b_D(K_{n,m}) = 1$.

$$(ii) \quad b_D(P_n) = \begin{cases} 0 & n = 2 \\ 2 & n = 3 \\ 1 & n = 4 \\ 2 & n \geq 5. \end{cases}$$

$$(iii) \quad b_D(C_n) = \begin{cases} 1 & n = 3, 4, 5 \\ 3 & n \geq 6. \end{cases}$$

Theorem 4.2 For every natural number k and $i \in \{1, \dots, k+1\}$, there exists a graph G_i such that $D(G_i) = 2k$ and $b_D(G_i) = i$. Also, there exists a graph H_i such that $D(H_i) = 2k+1$ and $b_D(H_i) = i$.

Proof. With respect to Theorem 2.6, we know that for every $i \in \{1, \dots, k+1\}$ there exists n_i such that $D(F_{n_i}) = 2k$. Also it can be concluded that there exist $2k-1$ consecutive friendship graphs with the distinguishing number $2k$. Without loss of generality, we assume that $D(F_{n_i}) = D(F_{n_i+1}) = \dots = D(F_{n_i+2k-2}) = 2k$. We claim that

$$(i) \quad b_D(F_{n_i+j}) = j+1 \text{ for } 0 \leq j \leq k-1, \text{ and}$$

$$(ii) \quad b_D(F_{n_i+k+j}) = k+1 \text{ for } 0 \leq j \leq k-2.$$

Let x_0 be the central vertex and x_{2t-1} and x_{2t} ($t \geq 1$) be the two adjacent vertex on the base of t -th triangle of the friendship graph.

(i) If we remove the edges $x_0x_1, x_0x_3, \dots, x_0x_{2j+1}$ from F_{n_i+j} , then F_{n_i+j} converted to the friendship graph F_{n_i-1} such that the end vertex of $j+1$ paths of order three identified to the central vertex of F_{n_i-1} . It can be computed that $D(F_{n_i+j} - x_0x_1 - x_0x_3 - \dots - x_0x_{2j+1}) = 2k-1$, and hence $b_D(F_{n_i+j}) \leq j+1$. If we remove less than $j+1$ edges from F_{n_i+j} , say e_1, \dots, e_l , $l \leq j$, then $F_{n_i+j} - e_1 - \dots - e_l$ has F_t where $t \geq n_i$ as its fixed induced subgraph, and hence $D(F_{n_i+j} - e_1 - \dots - e_l) = 2k$, so $b_D(F_{n_i+j}) = j+1$.

(ii) If we remove the edges $x_1x_2, x_3x_4, \dots, x_{2k-1}x_{2k}$ from F_{n_i+k+j} , then we obtain a graph such that it has made by identifying the central vertices of the star graph $K_{1,2k}$ and F_{n_i+j-1} , and so $D(F_{n_i+k+j} - x_1x_2 - x_3x_4 - \dots - x_{2k-1}x_{2k}) = 2k-1$. Hence $b_D(F_{n_i+k+j}) \leq k+1$. If we remove less than $k+1$ edges from F_{n_i+k+j} , say e_1, \dots, e_l , $l \leq k$, then $F_{n_i+k+j} - e_1 - \dots - e_l$ has F_{n_i+j+t} where $t \geq 0$ as its fixed induced subgraph, and hence $D(F_{n_i+k+j} - e_1 - \dots - e_l) = 2k$. Therefore $b_D(F_{n_i+k+j}) = k+1$.

To prove the second part of theorem, it is sufficient to note that there exists m_i such that $D(F_{m_i}) = D(F_{m_i+1}) = \cdots = D(F_{m_i+2k-1}) = 2k + 1$. By a similar argument we can show that $b_D(F_{m_i+j}) = j + 1$, and $b_D(F_{m_i+k+j}) = k + 1$ for $0 \leq j \leq k - 1$. So we have the result. \square

Similar to Proposition 3.1, we can prove $b_D(G) \leq b_D(G - e) + 1$. Now we can state some upper bonds for $b_D(G)$.

Theorem 4.3 *Let G be a simple graph of order $n \geq 2$ and size m with diameter d and maximum degree $\Delta \geq 3$. Then we have*

(i) $b_D(G) \leq m$.

(ii) $b_D(G) \leq m - \Delta + 1$.

(iii) For $n \geq d + 3$, $b_D(G) \leq m - d + 1$, and for $n < d + 3$, $b_D(G) \leq m - d + 2$.

(iv) If $\omega(G)$ is the clique number of G , then

$$b_D(G) \leq \begin{cases} m - \binom{\omega(G)}{2} + 1 & \text{if } n \leq 2\omega(G), \\ b_D(G) \leq m - \binom{\omega(G)}{2} + \omega(G) - 1 & \text{if } n > 2\omega(G). \end{cases}$$

Proof. Using $b_D(G) \leq b_D(G - e) + 1$ and mathematical induction, we have $b_D(G) \leq b_D(G - e_1 - \cdots - e_s) + s$ where $1 \leq s \leq m - 1$ and $m = |E(G)|$. To prove each case, it is sufficient to remove the edges of G until the connected components of size greater than or equal one has been stated in proof of each parts, appears. Next using $b_D(G) \leq b_D(G - e_1 - \cdots - e_s) + s$ and the value of $b_D(G - e_1 - \cdots - e_s)$, we can have the result.

- (i) The connected component of size greater than or equal one is the path P_2 , so $b_D(G) \leq b_D(P_2 \cup (n-2)K_1) + m - 1$. Since $b_D(P_2 \cup (n-2)K_1) = 1$ so $b_D(G) \leq m$.
- (ii) The connected component of size greater than or equal one is $K_{1,\Delta}$, so $b_D(G) \leq b_D(K_{1,\Delta} \cup (n - \Delta - 1)K_1) + m - \Delta$. Since $b_D(K_{1,\Delta} \cup (n - \Delta - 1)K_1) = 1$, the result follows.
- (iii) The connected component of size greater than or equal one is P_{d+1} , so $b_D(G) \leq b_D(P_{d+1} \cup (n - d - 1)K_1) + m - d$. If $n \geq d + 3$, then $b_D(P_{d+1} \cup (n - d - 1)K_1) = 1$, and if $n < d + 3$, then $b_D(P_{d+1} \cup (n - d - 1)K_1) \leq 2$. Hence the result follows.
- (iv) The connected component of size greater than or equal one is $K_{\omega(G)}$, so $b_D(G) \leq b_D(K_{\omega(G)} \cup (n - \omega(G))K_1) + m - \binom{\omega(G)}{2}$. If $n \leq 2\omega(G)$, then $b_D(K_{\omega(G)} \cup (n - \omega(G))K_1) = 1$, and if $n > 2\omega(G)$, then $b_D(K_{\omega(G)} \cup (n - \omega(G))K_1) = \omega(G) - 1$. Hence the result follows. \square

In 1956, Nordhaus and Gaddum obtained the lower and upper bounds for the sum of the chromatic numbers of a graph and its complement (actually, the upper bound was first proved by Zykov [7] in 1949). Since then, Nordhaus-Gaddum type bounds were obtained for many graph invariants. An exhaustive survey is given in [4]. Here, we give Nordhaus-Gaddum type inequalities for the bondage number on the distinguishing number of a graph.

Lemma 4.4 *If G is a simple connected graph with an edge e such that $b_D(G) \leq b_D(G - e)$ and $b_D(\overline{G} + e) \leq b_D(\overline{G})$, then $|b_D(G) - b_D(\overline{G})| \leq 1$.*

Proof. The proof is by induction on m , the number of edges of G . It can easily be seen that it is true for small value of m . So suppose that the result is true for graphs with size less than m . By $b_D(G) \leq b_D(G - e) + 1$, we can obtain $|b_D(G) - b_D(\overline{G})| \leq |b_D(G - e) - b_D(\overline{G} + e)|$, and so using $b_D(\overline{G - e}) = b_D(\overline{G} + e)$ and induction hypothesis we get the result. \square

Theorem 4.5 *If G is a simple graph with an edge e such that $b_D(G) \leq b_D(G - e)$ and $b_D(\overline{G} + e) \leq b_D(\overline{G})$, then*

$$1 \leq b_D(G) + b_D(\overline{G}) \leq 2 \min\{b_D(G), b_D(\overline{G})\} + 1.$$

Proof. This follows directly from Lemma 4.4. \square

Note that the upper bound of Theorem 4.5 is sharp and is achieved for example by the family of cycle graphs C_n with $n \geq 7$. In fact, if v_1, \dots, v_n are consecutive vertices of C_n , then $\overline{C_n} - \{v_1, v_{n-1}\} - \{v_2, v_{n-3}\}$ is an asymmetric graph, and so $b_D(\overline{C_n}) = 2$. About the sharpness of lower bound, it is sufficient to consider the complete graphs.

We conclude this section by mentioning a relation between the edge and vertex stability distinguishing number.

Theorem 4.6 *Let G be a graph. If there is an edge $e \in E(G)$ which is incident to one of the vertices of G such that their removal change the distinguishing number and $b_D(G) \leq b_D(G - e)$, then $b_D(G) \leq \left\lceil \frac{st_D(G)}{2} \right\rceil + 1$.*

Proof. If there exists an edge e such that $D(G) \neq D(G - e)$, then $b_D(G) = 1$, and so we have the result. If $D(G) = D(G - e)$ for all edges of G , then by induction on the size of G , we can write

$$b_D(G) \leq b_D(G - e) \leq \left\lceil \frac{st_D(G - e)}{2} \right\rceil + 1 \leq \left\lceil \frac{st_D(G)}{2} \right\rceil + 1.$$

where e is incident to one of the vertices of G such that their removal change the distinguishing number. Hence it can be concluded that $st_D(G - e) \leq st_D(G)$ for such edges. \square

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